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Use of Coles' Universal Wake Function for Compressible Turbulent Boundary Layers

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A two-parameter profile representation based on the law-of-the-wake and the law-of-the-wall is proposed for isoenergetic compressible turbulent boundary layers. The method of least squares was used to fit the proposed profile to experimental boundary-layer velocity profiles for a variety of flows including a normal shock induced separation, an oblique shock reflection and a flat-plate flow. Results indicate that for cases in which there is a significant departure of the boundary layer from flat-plate flow the proposed "wall-wake" profile provides a substantial improvement over the power-law representation of the velocity distribution. For both the normal shock induced separation and the oblique shock reflection, the proposed profile provided a good representation of the actual flow in the redeveloping region downstream of the interaction, indicating that the wall-wake representation should be useful in integral analysis of such flows. For all cases considered the two parameters of the profile, the boundary-layer thickness and the skin friction were found to be physically realistic.

Nomenclature

A = function of M and wall temperature, $\{[(\gamma - 1)/2] M_e^2/(T_w/T_e)\}^{1/2}$
B = function of M and wall temperature, $\{(1 + [(\gamma - 1)/2] M_e^2)/(T_w/T_e)\} - 1$
C = constant in law of the wall (usually 5.1)
C_f = skin-friction coefficient
K = constant in law of the wall (usually 0.4)
M = Mach number
Re_δ = Reynolds number based on δ
u = velocity in streamwise direction
u^* = Van Driest's generalized velocity, Eq. (6)
u_τ = friction velocity $(\tau_w/\rho_w)^{1/2}$
W = Coles' tabulated universal wake function
y = coordinate normal to wall
γ = ratio of specific heats
δ = boundary-layer thickness
μ = viscosity
ν = kinematic viscosity
π = coefficient of wake function
ρ = density
σ = $[(\gamma - 1)/2]M_e^2/\{1 + [(\gamma - 1)/2]M_e^2\}$
τ = shear stress

Subscripts

e = conditions at the edge of the boundary layer
w = conditions at the wall

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1. Introduction

FOR incompressible boundary layers, either with or without pressure gradient, Coles' Universal Wake Function,[1] when combined with the "law of the wall," provides a good representation of turbulent boundary-layer velocity profiles.

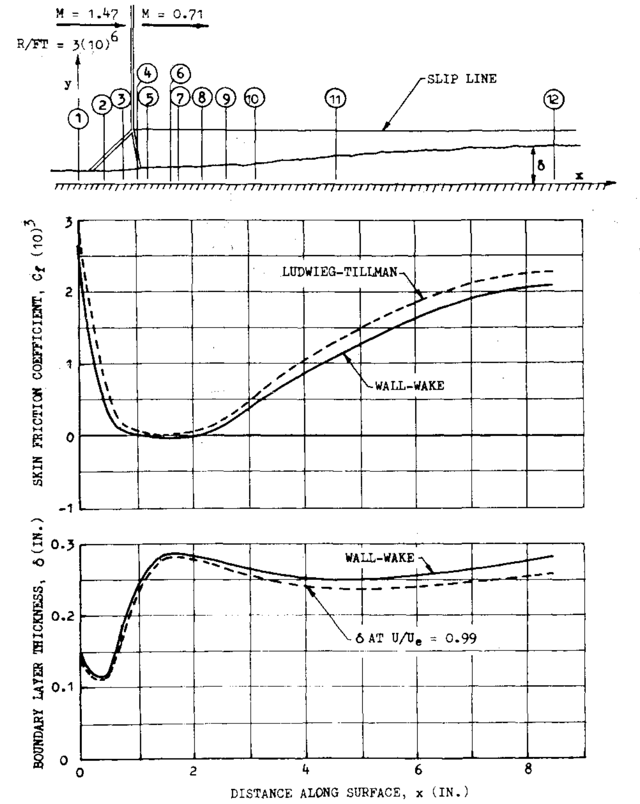

Fig. 1 δ and C_f from least squares fit of wall-wake profile to Seddon data.

Green[2] has used this velocity profile to represent fully developed, separated and redeveloping boundary layers in his integral analysis of two-dimensional incompressible base flows and suggests that it should be possible to extend this type of analysis to other separated flows, including compressible flow. For compressible flow, however, a similar profile equation which adequately represents the boundary layer in both the wall region and the wake region is not yet well established.

Maise and McDonald[3] have proposed an extension of the wall-wake velocity profile representation, utilizing the generalized velocities of Van Driest,[4] to compressible boundary-layer flows without heat transfer or pressure gradient and have demonstrated that this profile provides a good correlation with existing zero pressure gradient data.

The purpose of this paper is to suggest a general form of the wall-wake profile and to demonstrate that it provides a good representation of compressible turbulent boundary layers under such extreme conditions as those encountered within and downstream of both oblique and normal shock wave-boundary-layer interactions, as well as in zero pressure gradient flows.

2. Velocity Profile

For incompressible flow, the wall-wake velocity profile is given by

$$u/u_\tau = (1/K)\ln(yu_\tau/\nu) + C + (\pi/K)W(y/\delta) \quad (1)$$

where K and C are empirical constants and the profile parameter π is, in general, a function of the distance along a surface. π may be related to u_τ and δ in Eq. (1) by setting $u = u_e$ at $y = \delta$. From Eq. (1),

$$u_e/u_\tau = (1/K)\ln(\delta u_\tau/\nu) + C + (\pi/K)W(1) \quad (2)$$

where $W(1) = 2$. Subtracting Eq. (1) from Eq. (2), the wall-wake velocity profile may be rewritten as

$$u_e/u_\tau - u/u_\tau - (1/K)\ln(y/\delta) + (\pi/K)(2 - W) \quad (3)$$

For flat-plate flows, $\pi/K \simeq 1.25$ so Eq. (3) takes the form

$$u_e/u_\tau - u/u_\tau = -(1/K)\ln(y/\delta) + 1.25\,(2 - W) \quad (4)$$

Introducing compressibility into the differential equations of continuity, momentum and energy for turbulent flow, Van Driest[4] was able to obtain a law of the wall for compressible turbulent boundary layers of the form

$$u_e^*/u_\tau - u^*/u_\tau = -(1/K)\ln(y/\delta) \quad (5)$$

where

$$u^* = u_e(1/A)\arcsin\{[2A^2(u/u_e) - B]/(B^2 + 4A^2)^{1/2}\} \quad (6)$$

which for isoenergetic flows simplifies to

$$u^* = u_e(1/\sigma^{1/2})\arcsin\sigma^{1/2}(u/u_e) \quad (7)$$

Although Van Driest's profile did not include a wake component, Maise and McDonald noted that, for zero pressure gradient compressible boundary layers, when u^*/u_τ was plotted vs $\ln(y/\delta)$, the deviation from Van Driest's law of the wall in the outer portion of the layer was similar to the deviation from the law of the wall in an incompressible boundary layer. This led them to postulate for a compressible adiabatic boundary layer with zero pressure gradient that,

$$u_e^*/u_\tau - u^*/u_\tau = -(1/K)\ln(y/\delta) + 1.25\,(2 - W) \quad (8)$$

This is seen to be identical in form to the incompressible velocity defect law of Eq. (4) but with the effects of compressibility taken into account by replacing u by u^*. Using Eq. (8) and the isoenergetic form for u^*, they were able, for adiabatic zero pressure gradient flows, to correlate on a single curve boundary-layer profiles for a range of Reynolds numbers and for Mach numbers from 0 to 5.

Maise's and McDonald's success with the wall-wake profile suggests that a general profile similar in form to that given by Eq. (1) but with u replaced by u^*, might be applicable for compressible flows with pressure gradients. If this is true Eq. (1) then takes the form

$$u^*/u_\tau = (1/K)\ln(yu_\tau/\nu_w) + C + \pi W(y/\delta)/K \quad (9)$$

At the edge of the boundary layer, Eq. (9) becomes

$$u_e^*/u_\tau = (1/K)\ln(\delta u_\tau/\nu_w) + C + 2(\pi/K) \quad (10)$$

If, as in the general incompressible case, Eq. (9) is subtracted from Eq. (10), then

$$u_e^*/u_\tau - u^*/u_\tau = -(1/K)\ln(y/\delta) + (\pi/K)(2 - W) \quad (11)$$

where from Eq. (10),

$$\pi/K = \tfrac{1}{2}\{(u_e^*/u_\tau) - [(1/K)\ln(\delta u_\tau/\nu_w) + C]\} \quad (12)$$

Since we are considering isoenergetic flows, u^* in Eq. (11) is given by Eq. (7). If it is assumed that $\mu_w/\mu_e = (T_w/T_e)^{0.76}$, then for isoenergetic flows

$$\nu_w = \nu_e[1/(1 - \sigma)]^{1.76} \quad (13)$$

From Eqs. (7) and (11) we may now write

$$\frac{u}{u_e} = \frac{1}{\sigma^{1/2}}\sin\left\{\arcsin\sigma^{1/2}\left[1 + \frac{1}{K}\frac{u_\tau}{u_e^*}\ln(y/\delta) - \frac{\pi}{K}\frac{u_\tau}{u_e^*}\left(1 + \cos\frac{\pi y}{\delta}\right)\right]\right\} \quad (14)$$

where for mathematical convenience[3] the quantity $(2 - W)$ has been replaced by the essentially equivalent quantity $1 + \cos(\pi y/\delta)$. From Eqs. (12) and (13) and the definitions of u_τ, u_e^*, and C_f, the parameters u_τ/u_e^* and π/K occurring in Eq. (14) are

$$u_\tau = u_e^* = [(C_f/2)\,\sigma/(1 - \sigma)]^{1/2}/\arcsin\sigma^{1/2} \quad (15)$$

and

$$\frac{\pi}{K} = \frac{1}{2}\left\{\frac{1}{u_\tau/u_e^*} - \frac{1}{K}\ln\left[Re_\delta\left(\frac{C_f}{2}\right)^{1/2}(1-\sigma)^{1.26}\right] - C\right\} \quad (16)$$

Equation (14), when combined with Eqs. (15) and (16), gives a velocity profile representation which depends only on the Mach number M_e, the skin-friction coefficient C_f, and a Reynolds number Re_δ. The following section will show the suitability of this profile for representing experimental velocity profiles for a variety of compressible turbulent boundary-layer flows.

Comparisons with Experiment

The method of least squares was used to fit the wall-wake profile, Eq. (14), to a number of experimental velocity profiles for a range of Mach numbers and flow conditions. The values of C_f and δ determined by the curve fit were compared, respectively, with Ludwieg-Tillman C_f values and the values of $\delta_{0.99}$ corresponding to $u/u_e = 0.99$ from the data.

Three examples are presented. The first is a study of the interaction of a normal shock wave with a boundary layer at $M = 1.47$ by Seddon.[5] The second is a study by Seebaugh[6] of an interaction between a conical shock wave and an axially symmetric boundary layer at $M = 3.78$. An investigation by Rubesin et al.[7] of a turbulent boundary layer on a flat plate at $M = 2.45$ is the final case considered.

Seddon presented 12 boundary-layer velocity profiles obtained in the vicinity of the shock impingement. As shown in Fig. 1, these profiles were obtained in the undisturbed upstream flow, in the separated region, and in the reattaching and redeveloping regions at the downstream end of the interaction. Figure 2 shows comparisons of the least squares fit of the wall-wake profile to Seddon's velocity profiles.

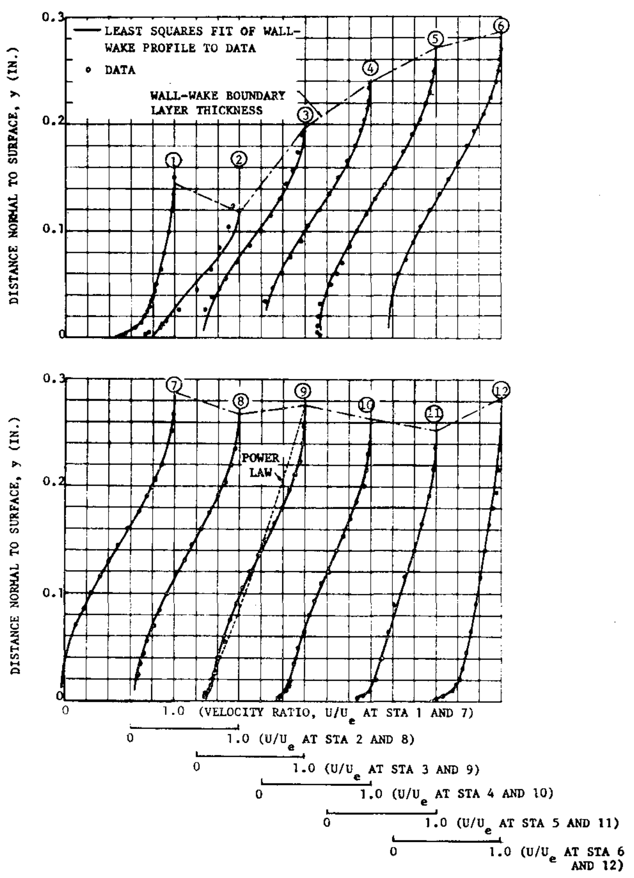

Fig. 2 Velocity profiles, Seddon data.

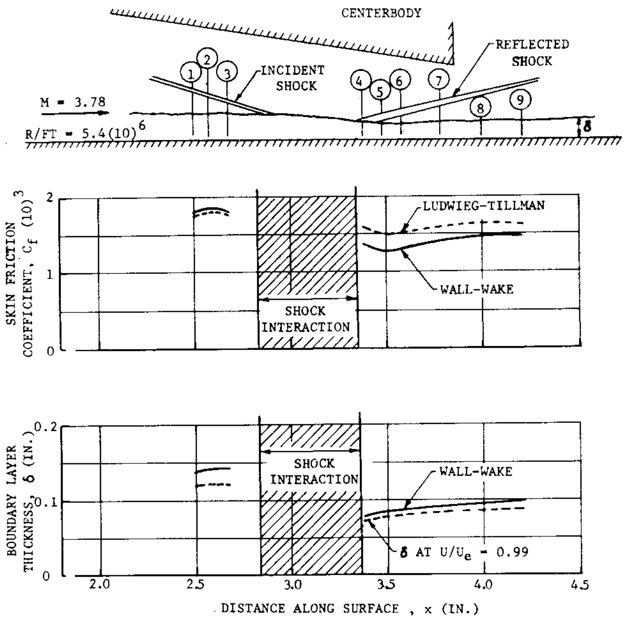

Fig. 3 δ and C_f from least squares fit of wall-wake profile to Seebaugh data.

Except for profiles 2 and 3, which were near separation, the experimental results are well represented by the wall-wake profile. For purposes of comparison a power law representation of profile 9 has been shown.

A comparison of the wall-wake and the Ludwieg-Tillman values of C_f for the Seddon data is shown in Fig. 1. The Ludwieg-Tillman C_f values were calculated using the compressibility transformation procedure described by Sasman and Cresci[8] to extend the incompressible Ludwieg-Tillman formulation to compressible flow. The values of momentum thickness and shape factor required to calculate the Ludwieg-Tillman C_f were obtained from the integrated mass and momentum fluxes of the least squares fit of the wall-wake profiles to the experimental velocity profiles. The wall-wake values of C_f were within 10% of the calculated Ludwieg-Tillman values over most of the region of interaction. The Ludwieg-Tillman formulation is known to be inaccurate near separation and in the region of flow that Seddon indicated was separated, the wall-wake C_f values were slightly negative while the Ludwieg-Tillman values were slightly positive. Thus, in this region the wall-wake C_f values should be more realistic than the Ludwieg-Tillman values.

A comparison also was made between the wall-wake values for δ and the values of $\delta_{0.99}$ for the Seddon data. The results are plotted in Fig. 1. The wall-wake values were found to be within 5% of the experimental values for most of the interaction. It should be noted that although δ corresponding to $u/u_e = 0.99$ is widely used for incompressible boundary layers, the same cannot be said for compressible flows. For example, as reported by Baronti and Libby,[9] δ corresponding to a value of $u/u_e = 0.995$ has been used by several investigators to describe turbulent boundary layers at higher Mach numbers.

In the interaction studied by Seebaugh, the conical shock wave was generated by a 10° half-angle cone. As is shown in Fig. 3 three profiles are presented for stations upstream of the shock impingement point and five for stations downstream of the reflected shock. The least squares fits of the wall-wake profile to Seebaugh's velocity profile data are shown in Fig. 4. The velocity profile fits are good for all of these profiles.

The wall-wake values of C_f for the Seebaugh data are compared with the Ludwieg-Tillman values in Fig. 3. The values agreed within 5% upstream of the interaction and 15% downstream. A comparison between the wall-wake values of

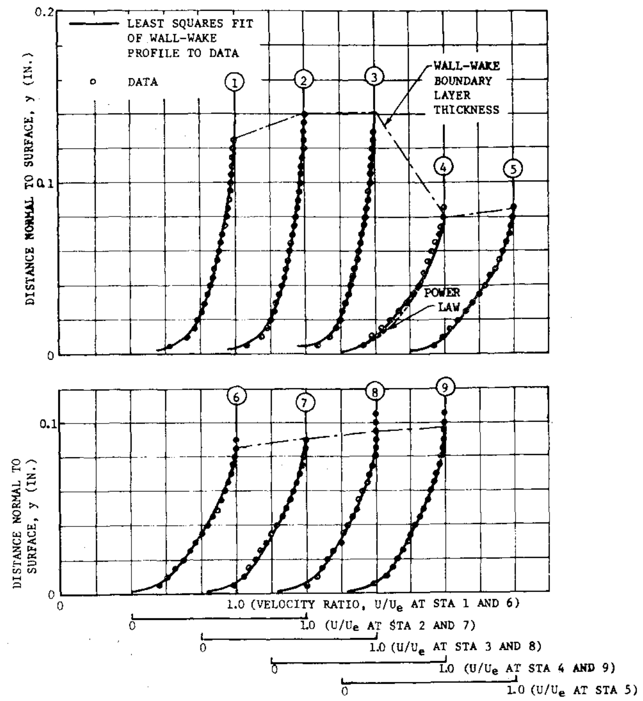

Fig. 4 **Velocity profiles, Seebaugh data.**

δ and $\delta_{0.99}$ is also made in Fig. 3. The agreement was within 15% upstream of the interaction and 12% downstream.

Rubesin, Maydew, and Varga[7] studied the development of a turbulent boundary layer on a flat plate at $M = 2.45$. Velocity profiles were reported for the five wall locations shown in Fig. 5. The wall-wake values of C_f were within 10% of the Ludwieg-Tillman values and the wall-wake values of the boundary-layer thickness were within 15% of the $\delta_{0.99}$ values for the five profiles considered.

The least squares fits of the wall-wake profile to the Rubes'n data are shown in Fig. 6, and again, are found to provide a good representation of the velocity distribution.

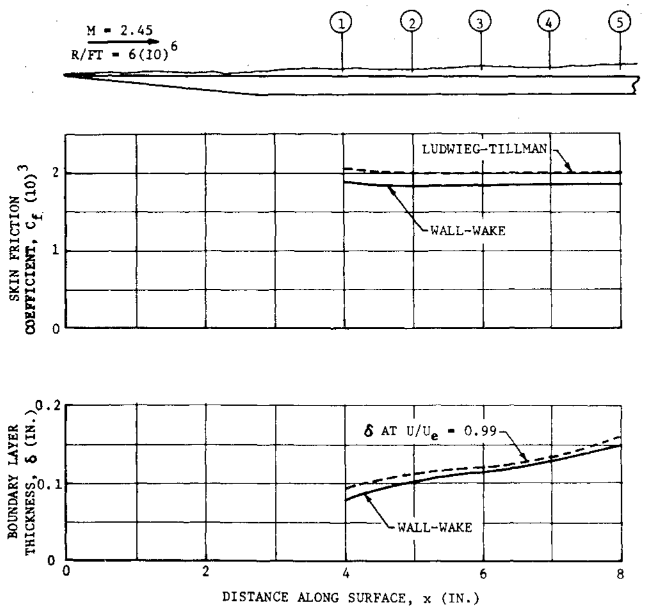

Fig. 5 **δ and C_f from least squares fit of wall-wake profile to Rubesin, Maydew, and Varga data.**

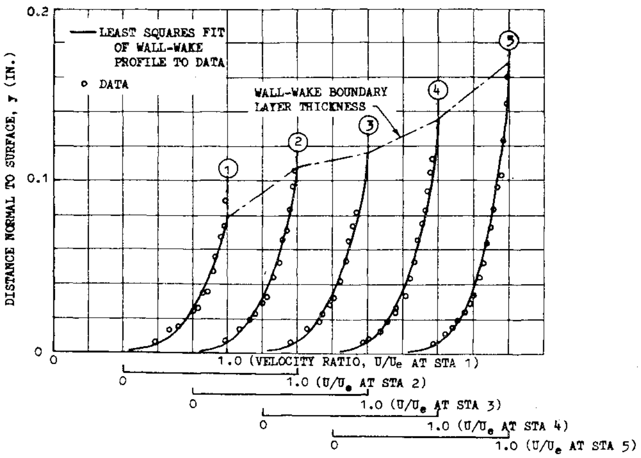

Fig. 6 **Velocity profiles, Rubesin, Maydew, and Varga data.**

Conclusions

An extension of the law-of-the-wall, law-of-the-wake two-parameter velocity profile representation of boundary-layer flows to isoenergetic compressible flow has been proposed. This profile has been found to provide a good representation of experimental velocity data for a wide variety of flow conditions, including fully developed flat-plate flows, flows in separated and reattaching regions, and flows in an unseparated oblique shock wave boundary-layer interaction region.

The two parameters of the proposed wall-wake profile, i.e., C_f and δ calculated by a least squares fit of the profile to a given experimental velocity profile appear to be physically realistic. The results of this study indicate that this profile representation should be useful in integral analyses of problems such as shock wave-boundary-layer interactions.

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